

THE DUFFIN-KEMMER-PETIAU OSCILLATOR

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Abstract

In view of current interest in relativistic spin-one systems and the recent work on the Dirac Oscillator, we introduce the Duffin-Kemmer-Petiau (DKP) equation obtained by using an external potential linear in \mathbf{r} . Since, in the non-relativistic limit, the spin 1 representation leads to a harmonic oscillator with a spin-orbit coupling of the Thomas form, we call the equation the DKP oscillator. This oscillator is a relativistic generalisation of the quantum harmonic oscillator for scalar and vector bosons. We show that it conserves total angular momentum and that it is *exactly* solvable. We calculate and discuss the eigenspectrum of the DKP oscillator in the spin 1 representation.

1 The DKP Oscillator

The focus of attention in this paper is to generalise the concept of the quantum harmonic oscillator to relativistic vector bosons.

For a free scalar or vector boson of mass m , the relativistic DKP equation [1] is

$$(c\boldsymbol{\beta} \cdot \mathbf{p} + mc^2)\psi = i\hbar\beta^0 \frac{\partial\psi}{\partial t} \quad (1.1)$$

where the internal variables β^μ ($\mu = 0, 1, 2, 3$) satisfy the commutation relation

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\nu\lambda}\beta^\mu. \quad (1.2)$$

In the spin 1 representation, the β^μ are 10×10 matrices while the dynamical state ψ is a ten-component spinor.

For the external potential which we introduce with the non-minimal substitution

$$\mathbf{p} \longrightarrow \mathbf{p} - im\omega\eta^0\mathbf{r}, \quad (1.3)$$

where ω is the oscillator frequency and $\eta^0 = 2\beta^{02} - 1$, the DKP equation for the system is

$$[c\boldsymbol{\beta} \cdot (\mathbf{p} - im\omega\eta^0\mathbf{r}) + mc^2]\psi = i\hbar\beta^0 \frac{\partial\psi}{\partial t}. \quad (1.4)$$

This external potential, which is of Lorentz tensor type, does not conserve the orbital and spin angular momenta, since

$$[\beta\eta^0 \cdot \mathbf{r}, \mathbf{L}] = -i(\beta\eta^0 \wedge \mathbf{r}) \quad \text{and} \quad [\beta\eta^0 \cdot \mathbf{r}, \mathbf{S}] = i(\beta\eta^0 \wedge \mathbf{r}), \quad (1.5)$$

but it does conserve the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

In the spin 1 representation of eq.(4), the dynamical state ψ is chosen as the 10-component spinor

$$\psi(\mathbf{r}) = \begin{pmatrix} i\varphi \\ \mathbf{A}(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}) \\ \mathbf{C}(\mathbf{r}) \end{pmatrix} \quad \text{with} \quad \mathbf{A} \equiv \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \quad \mathbf{B} \equiv \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \quad \text{and} \quad \mathbf{C} \equiv \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \quad (1.6)$$

so that, for stationary states, the equation of motion eq.(4) decomposes into

$$\begin{cases} mc^2\varphi = ic\mathbf{p}^- \cdot \mathbf{B} \\ mc^2\mathbf{A} = E\mathbf{B} - c\mathbf{p}^+ \wedge \mathbf{C} \\ mc^2\mathbf{B} = E\mathbf{A} + ic\mathbf{p}^+\varphi \\ mc^2\mathbf{C} = -c\mathbf{p}^- \wedge \mathbf{A} \end{cases} \quad (1.7)$$

where $\mathbf{p}^\pm = \mathbf{p} \pm im\boldsymbol{\omega}\mathbf{r}$. Since \mathbf{A} is the 3-component spinor analogous to the Dirac upper component, we seek the wave equation for \mathbf{A} . It is straightforward to eliminate φ , \mathbf{B} and \mathbf{C} in favor of \mathbf{A} so that one gets

$$(E^2 - m^2c^4)\mathbf{A} = [c^2(\mathbf{p}^2 + m^2\omega^2\mathbf{r}^2) - 3\hbar\omega mc^2 - 2\hbar\omega mc^2\mathbf{L} \cdot \mathbf{s}]\mathbf{A} - \frac{1}{m^2}\mathbf{p}^+ \{\mathbf{p}^- \cdot [\mathbf{p}^+ \wedge (\mathbf{p}^- \wedge \mathbf{A})]\} \quad (1.8)$$

where \mathbf{L} is the orbital angular momentum and \mathbf{s} the 3×3 spin one operator. In the non-relativistic limit $\varepsilon \ll mc^2$, the fourth term in eq.(1.8) becomes negligible, since it is of order $1/m^3$, so that the wave equation for \mathbf{A} can be written

$$\varepsilon\mathbf{A} \simeq \left[\frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2 - \frac{3}{2}\hbar\omega - \hbar\omega\mathbf{L} \cdot \mathbf{s} \right]\mathbf{A} \quad (1.9)$$

which characterises the usual harmonic oscillator in addition to a spin-orbit coupling, absent for scalar DKP bosons, of strength $-\hbar\omega$. Note that the strength of this coupling is half the one obtained from the Dirac oscillator [2].

Since the spin 1 representation of eq.(1.4) leads to the usual three-dimensional (3D) oscillator, in the non-relativistic limit, we refer to the system it describes as the Duffin-Kemmer-Petiau oscillator.

2 Solution to the vector DKP oscillator problem

For the $S = 1$ central field problem, the general eigenfunction we use takes the form [3]

$$\psi_{JM}(\mathbf{r}) = \frac{1}{r} \begin{pmatrix} i\phi_{nJ}(r)Y_{JM}(\Omega) \\ \sum_L F_{nJL}(r)\mathbf{Y}_{JL1}^M(\Omega) \\ \sum_L G_{nJL}(r)\mathbf{Y}_{JL1}^M(\Omega) \\ \sum_L H_{nJL}(r)\mathbf{Y}_{JL1}^M(\Omega) \end{pmatrix}. \quad (2.1)$$

Putting ψ_{JM} into eq.(1.4) results in ten coupled radial differential equations which can be decoupled into two sets associated with $(-1)^J$ and $(-1)^{J+1}$ parities. We call the $(-1)^J$ solutions

natural-parity (or magnetic-like) states while we refer to the $(-1)^{J+1}$ solutions as unnatural-parity (or electric-like) states. With the notation

$$R_{nJJ}(r) = R_o \quad , \quad R_{nJJ\pm 1}(r) = R_{\pm 1}, \quad R \equiv F, G, H \quad (2.2)$$

the set associated with $(-1)^J$ parity is

$$EF_o = mc^2G_o \quad (2.3)$$

$$\hbar c \left(\frac{d}{dr} - \frac{J+1}{r} + \frac{m\omega r}{\hbar} \right) F_o = -\frac{1}{\zeta_J} mc^2 H_1 \quad (2.4)$$

$$\hbar c \left(\frac{d}{dr} + \frac{J}{r} + \frac{m\omega r}{\hbar} \right) F_o = -\frac{1}{\alpha_J} mc^2 H_{-1} \quad (2.5)$$

$$-\zeta_J \left(\frac{d}{dr} + \frac{J+1}{r} - \frac{m\omega r}{\hbar} \right) H_1 - \alpha_J \left(\frac{d}{dr} - \frac{J}{r} - \frac{m\omega r}{\hbar} \right) H_{-1} = \frac{1}{\hbar c} (mc^2 F_o - EG_o). \quad (2.6)$$

For unnatural parity states, the radial differential equations are coupled in the following way :

$$\hbar c \left(\frac{d}{dr} - \frac{J+1}{r} - \frac{m\omega r}{\hbar} \right) H_o = -\frac{1}{\zeta_J} (mc^2 F_1 - EG_1) \quad (2.7)$$

$$\hbar c \left(\frac{d}{dr} + \frac{J}{r} - \frac{m\omega r}{\hbar} \right) H_o = -\frac{1}{\alpha_J} (mc^2 F_{-1} - EG_{-1}) \quad (2.8)$$

$$-\zeta_J \left(\frac{d}{dr} + \frac{J+1}{r} + \frac{m\omega r}{\hbar} \right) F_1 - \alpha_J \left(\frac{d}{dr} - \frac{J}{r} + \frac{m\omega r}{\hbar} \right) F_{-1} = \frac{1}{\hbar c} mc^2 H_o \quad (2.9)$$

$$\hbar c \left(\frac{d}{dr} - \frac{J+1}{r} - \frac{m\omega r}{\hbar} \right) \phi = -\frac{1}{\alpha_J} (mc^2 G_1 - EF_1) \quad (2.10)$$

$$\hbar c \left(\frac{d}{dr} + \frac{J}{r} - \frac{m\omega r}{\hbar} \right) \phi = \frac{1}{\zeta_J} (mc^2 G_{-1} - EF_{-1}) \quad (2.11)$$

$$-\alpha_J \left(\frac{d}{dr} + \frac{J+1}{r} + \frac{m\omega r}{\hbar} \right) G_1 + \zeta_J \left(\frac{d}{dr} - \frac{J}{r} + \frac{m\omega r}{\hbar} \right) G_{-1} = \frac{1}{\hbar c} mc^2 \phi. \quad (2.12)$$

To obtain the exact solution for the magnetic-like states, we eliminate G_o, H_{\pm} in favor of F_o in eq.(2.6). This yields the eigenvalues [4]

$$\frac{1}{2mc^2} (E_{N,J}^2 - m^2 c^4) = (N+1)\hbar\omega \quad (2.13)$$

with the principal quantum number $N = 2n + J$ (n is the radial quantum number). Note that the oscillator levels are equidistant and degenerate; the zero-point energy differs here from the one we found for the scalar DKP bosons.

The exact eigenvalues of the radial equations associated with unnatural-parity states can be shown [4] to be

$$\frac{1}{2mc^2}(E_{\pm}^2 - m^2c^4) = (N + \frac{3}{2})\hbar\omega + J(J+1)\frac{(\hbar\omega)^2}{mc^2} \mp \Delta \quad (2.14)$$

where

$$\Delta = \hbar\omega \left(J + \frac{1}{2} \right) \left(1 + \frac{a_1}{a_0} \frac{\hbar\omega}{mc^2} + \frac{a_2}{a_0} \left(\frac{\hbar\omega}{mc^2} \right)^2 \right)^{\frac{1}{2}} \quad (2.15)$$

with $a_0 = (2J+1)^2$, $a_1 = 4J(J+1)(2N+3)$ and $a_2 = 4J^2(J+1)^2$ where N , a positive integer, is the principal quantum number.

As shown in eq.(23), the energy of the DKP oscillator in unnatural parity states involves the usual 3-dimensional harmonic oscillator energy, a second term proportional to $J(J+1)$ which appears as some kind of rotational energy and a third energy contribution Δ which is a complicated function of the oscillator frequency, J and N with no obvious physical interpretation.

In the limit where the oscillator frequencies are such that $\hbar\omega \ll mc^2$, keeping only the first-order term in ω in eqs.(2.14-15) leads to

$$\frac{1}{2mc^2}(E_+^2 - m^2c^4) \equiv \epsilon_{n,r.}^+ \simeq (N - J + 1)\hbar\omega \quad (2.16)$$

$$\frac{1}{2mc^2}(E_-^2 - m^2c^4) \equiv \epsilon_{n,r.}^- \simeq (N + J + 2)\hbar\omega \quad (2.17)$$

This is best illustrated in fig.1 which shows, for fixed values of N and J , the variations of the relativistic and non-relativistic eigen-energies with $\hbar\omega/mc^2$.

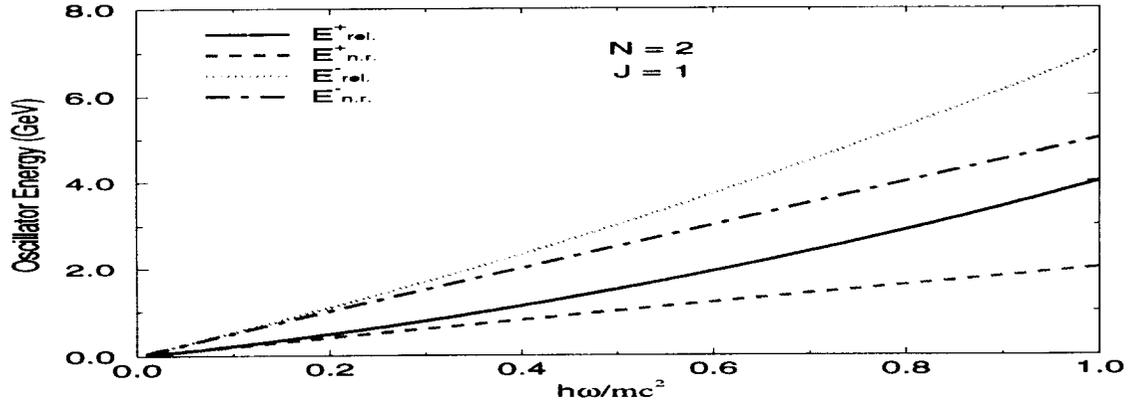


Figure 1: Variation of the DKP and non-relativistic oscillator energies with ω .

This shows that our solutions have the correct non-relativistic limits since the levels in eqs.(2.16-17) are those of a usual 3D non-relativistic oscillator with a spin-orbit coupling of strength $-\hbar\omega$. In this limit, they could have also been obtained directly from eq.(1.9). Furthermore, taking this limit suggests the interpretation of the E_+ and E_- energies as “spin-orbit partners”, E_+ being associated with $J = L + 1$ and E_- with $J = L - 1$.

The unnatural parity E_+ levels for $N \leq 9$ are presented in fig.2 alongside the $\epsilon_{n,r}^+$ and the $(N + \frac{3}{2})\hbar\omega$ levels for reference. For a given value of $N - J$, all the non-relativistic energy levels (N, J^π) (with $N - J$ odd and $J < N$) are infinitely degenerate. This accidental degeneracy is not present in the exact DKP oscillator E_+ eigenspectrum whose levels are found to cluster in several groups of states belonging to the same $N - J$ oscillator shell.

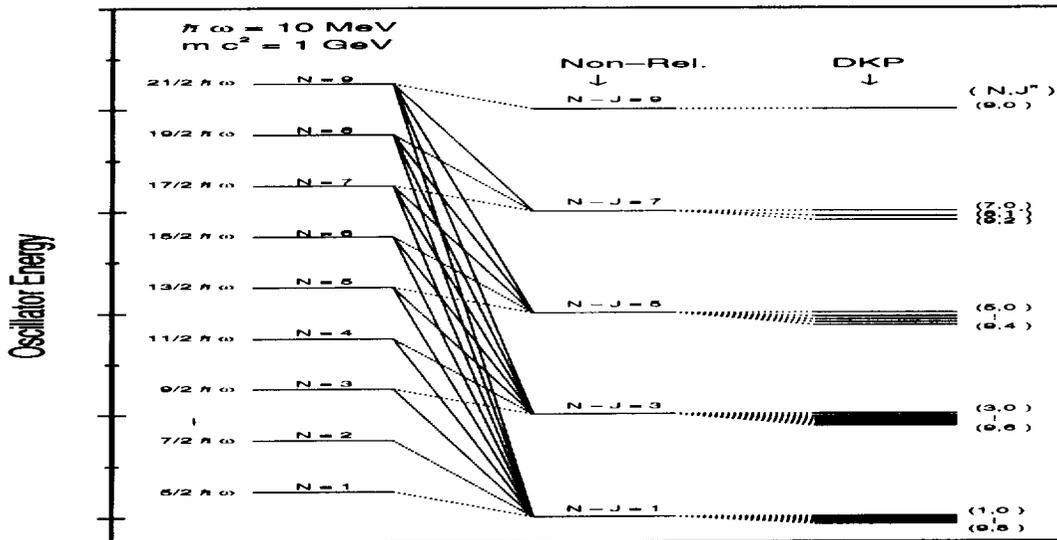


Figure 2: DKP and non-relativistic spectra associated with $J = L + 1$ for $N \leq 9$. The dotted lines between the DKP and non-relativistic oscillator levels link states with the same quantum numbers (N, J^π) .

The E_- eigenspectrum is now presented in fig.3 together with the non-relativistic $\epsilon_{n,r}^-$ energy levels for $N \leq 9$.

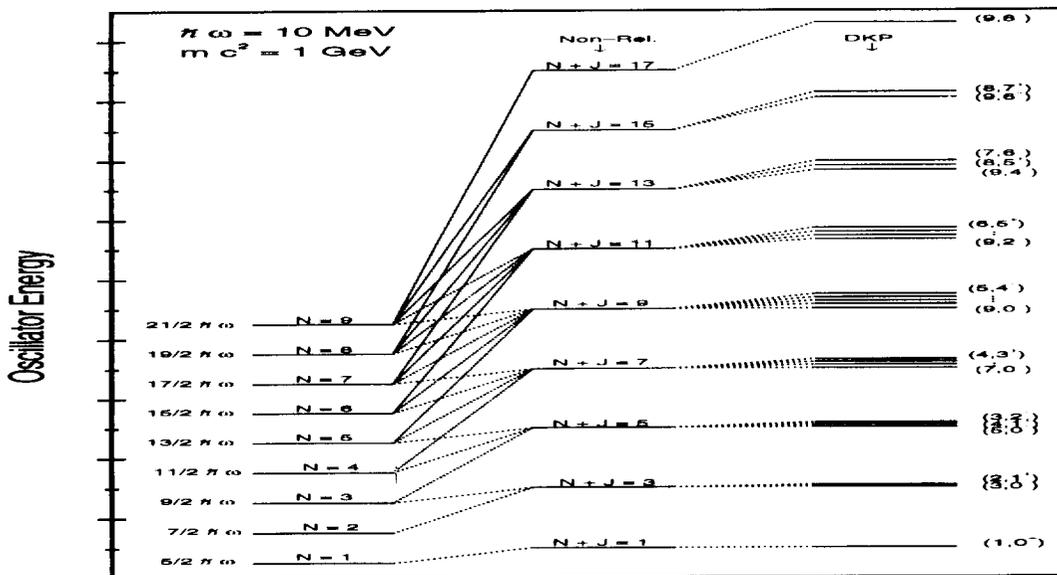


Figure 3: DKP and non-relativistic spectra associated with $J = L - 1$ for $N \leq 9$.

While all the non-relativistic (N, J^π) levels associated with the same $N + J$ oscillator shell are degenerate, with a finite degeneracy in this case, their relativistic analogues are not. The

exact DKP oscillator states are found to cluster into bands of states belonging to specific values of $N + J$.

For a more quantitative analysis of the E_- bands of the DKP oscillator spectrum, we plotted in fig.4 the E_- energy levels, belonging to the $N + J = 49$ band for instance, against $J(J + 1)$ for different oscillator frequencies.

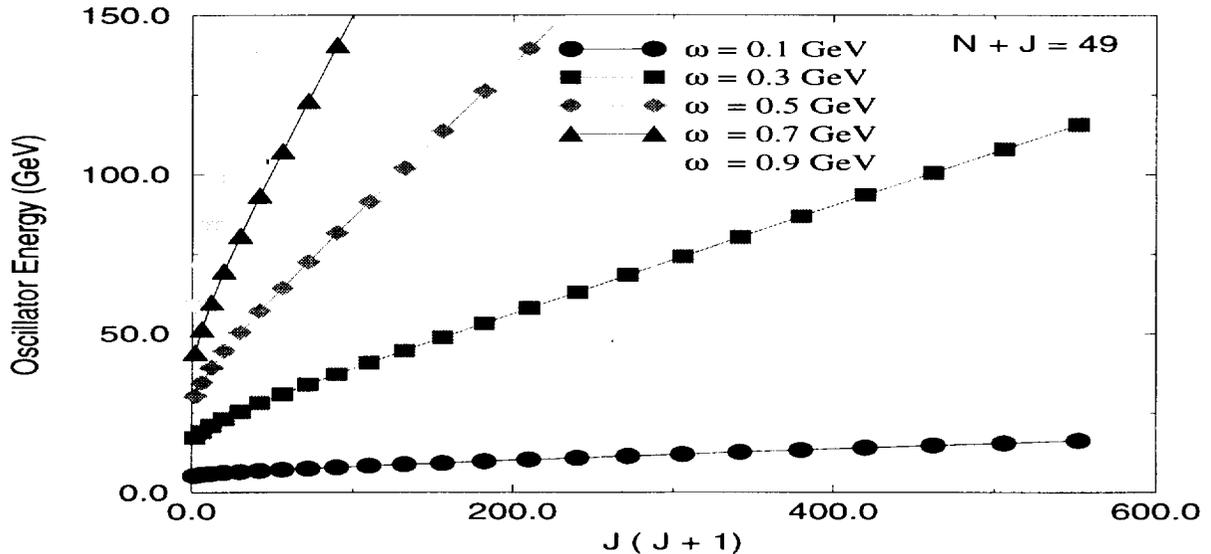


Figure 4: Energy levels of the $N + J = 49$ band as a function of $J(J + 1)$ for different oscillator frequencies.

It is indeed remarkable that the DKP oscillator energies constitute nearly perfect rotational bands. There are deviations from the rotational patterns at low angular momenta. These single particle rotational bands are of the finite type since for $N + J$ fixed they terminate at some J_{max} . The effective rotational moments of inertia are sensitive to the oscillator frequencies since the slopes of the bands are found to vary substantially with increasing ω .

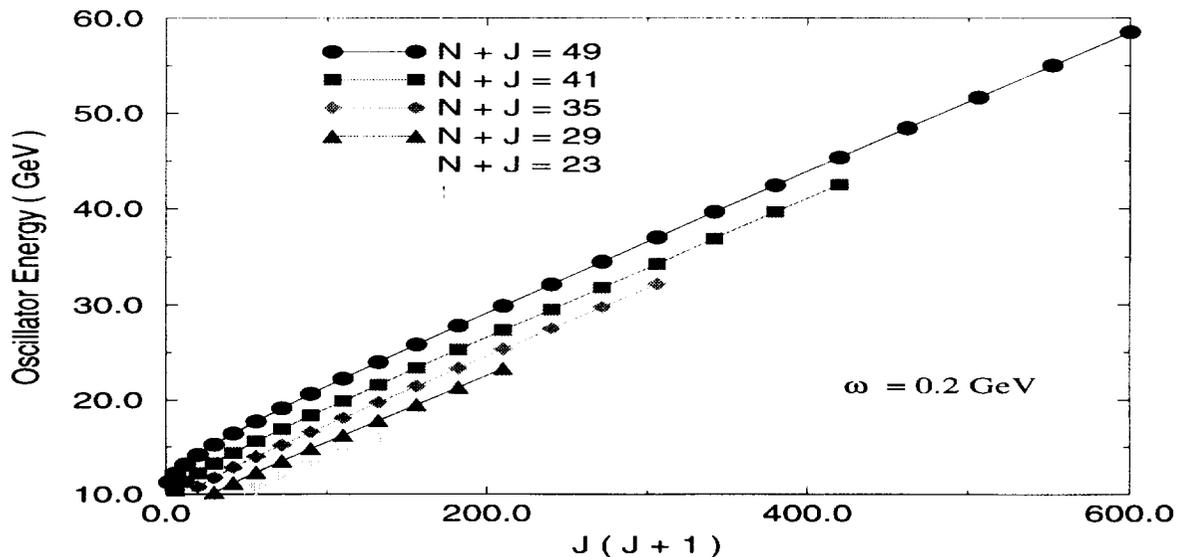


Figure 5: Energy levels of the $N + J = 23, 29, 35, 41, 49$ band as a function of $J(J + 1)$ for $\hbar\omega = 0.2\text{GeV}$.

Fig.5 alternately represents the energies of five different $N + J$ bands for a specific oscillator frequency as a function of $J(J + 1)$. The DKP oscillator energies now lie on rotational bands whose slopes hardly change with $N + J$ which implies that the effective rotational moments of inertia are rigid and insensitive to such variations.

Of course, it should be pointed out that these rotational bands are unlike the usual ones where the levels are associated with the same intrinsic motion but different angular momenta. Here the single particle states involve different radial as well as rotational motions. Note that this behaviour is not particularly tied to this DKP oscillator. Buck [5] also found that, when solving the Schrödinger equation for deep, bell-shaped potentials, the levels with a fixed value of $N = 2n + \ell$ (these states are degenerate for a harmonic oscillator) lie on a straight line when plotted against $\ell(\ell + 1)$. Geometric arguments in terms of the shapes of the potentials which can give rise to these rotational-like bands have been put forward to explain this behaviour [5][6].

3 Conclusion

We have introduced a new potential in the DKP equation. Since, in the non-relativistic limit, the DKP equation of motion leads to the usual harmonic oscillator with a spin-orbit coupling of the Thomas form, we call the system a DKP oscillator. This oscillator is a relativistic generalisation of the quantum harmonic oscillator for vector bosons. We have shown that it conserves the total angular momentum, that it is exactly soluble and we have computed and discussed its eigen-solutions.

The renewed interest in the Dirac oscillator has generated studies of its group theoretical properties [7] and hidden supersymmetry [8][9] among others. Such investigations of the DKP oscillator would be most useful to gain further insight into the physical meaning of this oscillator.

This study is on the other hand relevant to the work on relativistic equations for two fermions and particularly to those of Krokowski's type [10]. Since they tend to the DKP equation in the point-like limit of tightly bound-states, exact solutions of the latter may provide useful information about this class of relativistic two-body equations.

Acknowledgments

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